

METHOD OF CALCULATING THE FLOW IN A CIRCULAR NOZZLE

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16. Abstract A modification of the Newton-Kantorovich variational method is proposed for calculating flows in the subsonic and transonic portions of circular nozzles. The knowledge of these flows is necessary for solving the variational problem for the supersonic portion of the nozzle. The method proposed is shown to be as accurate as the method of characteristics, but to provide substantial savings in computer time.			
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Notation

a, b, β, t, r are the coefficients of the polynomials for the unknown functions, u is the axial velocity component, v is the velocity component which is normal to the nozzle axis, p is the pressure, ρ is the density, T is the temperature, θ is the tangent of the angle between the velocity vector and the nozzle axis, F, ϕ are variables in the coefficient equation, q is the flow density, k is the heat capacity ratio, r is the radius, n is the mean isentropic flow index.

Subscripts: k is the nozzle contour, cr is the critical cross section.

METHOD OF CALCULATING THE FLOW IN A CIRCULAR NOZZLE

Yu. M. Danilov

Currently, there is great interest in calculating gas flows /60* in the subsonic and transonic parts of nozzles with large gradients of the gas dynamic parameters in the direction which is normal to the axis [1]-[3].

In article [4], the direct problem of optimizing nozzles which operate on gas with condensate particles was formulated. The variational problem was solved numerically for the supersonic part of the nozzle. The parameters of the flow in the initial profile were assumed to be given. However, to completely solve the problem of optimizing the nozzle profile, the initial conditions for the method of characteristics used in [4] are the results obtained from calculating the flow in the subsonic and transonic parts of the nozzle. The use of methods [1]-[3] for this purpose is difficult, because of the excessive computer time needed to solve this problem. In this article, a modification of the Newton-Kantorovich method is proposed for calculating the flow in the subsonic part of the nozzle and for obtaining the initial parameter in the method of characteristics, which is specially tailored to the solution of the problem. While the accuracy of the results is the same, the machine time needed to solve the problem is reduced considerably in comparison with the solution obtained by the method described in [2].

The equations of gas dynamics are written in the form

* Numbers in the margin indicate pagination in the foreign text.

$$\begin{aligned}
& \frac{\partial}{\partial x} (\rho u r) + \frac{\partial}{\partial r} (\rho v r) + A_1 = 0, \\
& u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} + A_2 = 0, \\
& u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial r} + A_3 = 0, \\
& u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial r} - kRT \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial r} \right) + A_4 = 0.
\end{aligned} \tag{1}$$

The terms A_1 , A_2 , A_3 , and A_4 take into account the possibility that the suspended particles which are present in it can have an effect on the gas flow [5]. For a pure gas flow (the suspended particle concentration is zero) $A_i = 0$ ($i = 1, 2, 3, 4$).

The velocity is referred to the stagnation sound velocity /61 and the density, pressure, and temperature to the corresponding magnitudes in a gas at rest.

We consider the case of a flow regime when in some cross section, which lies in the most narrow region (the throat) of the nozzle, the velocity of the flow is equal to the local velocity of sound. It is known that in this case in the throat of the nozzle, the integral of the mass flux attains its maximum:

$$q / s_{cr} = \left[\frac{2}{r_{cr}^2} \int_0^{r_k} \rho u r dr \right]_{max}. \tag{2}$$

To facilitate the formulation of the boundary conditions, we make a change of variable. Instead of the independent variable r , we use the dimensionless ordinate

$$y = \frac{r}{r_k},$$

which is the ordinate referred to the ordinate of the contour of the nozzle in the cross section under consideration. Instead of the unknown v (the projection of the velocity on the r axis), we seek the tangent of the angle between the velocity vector and the nozzle axis:

$$\theta = \frac{v}{u}.$$

In the new variables, Eqs. (1) take on the form

$$\left. \begin{aligned} \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{1}{r_k} \left[\rho (\theta - y \theta_k) \frac{\partial u}{\partial y} + u (\theta - y \theta_k) \frac{\partial \rho}{\partial y} + \rho u \frac{\partial \theta}{\partial y} + \rho u \frac{\theta}{y} \right] + \\ + A'_1 = 0, \\ u \frac{\partial u}{\partial x} + \frac{1}{r_k} u (\theta - y \theta_k) \frac{\partial u}{\partial y} + \frac{1}{k} \left(\frac{\partial T}{\partial x} + \frac{T}{\rho} \cdot \frac{\partial \rho}{\partial x} \right) - \\ - \frac{1}{k r_k} y \theta_k \left(\frac{\partial T}{\partial y} + \frac{T}{\rho} \cdot \frac{\partial \rho}{\partial y} \right) + A'_2 = 0, \\ u \theta \frac{\partial u}{\partial x} + u^2 \frac{\partial \theta}{\partial x} + \frac{1}{r_k} \left[\theta u (\theta - y \theta_k) \frac{\partial u}{\partial y} + u^2 (\theta - y \theta_k) \frac{\partial \theta}{\partial y} \right] + \\ + \frac{1}{k r_k} \left(\frac{\partial T}{\partial y} + \frac{T}{\rho} \cdot \frac{\partial \rho}{\partial y} \right) + A'_3 = 0, \\ \frac{u}{k} \cdot \rho \cdot \frac{\partial T}{\partial x} + \frac{1}{r_k} \frac{\rho u}{k} (\theta - y \theta_k) \frac{\partial T}{\partial y} + \left[\frac{\partial \rho}{\partial x} + \frac{1}{r_k} (\theta - \theta_k y) \frac{\partial \rho}{\partial y} \right] \times \\ \times \left(\frac{1}{k} - 1 \right) T u + A'_4 = 0. \end{aligned} \right\} \quad (3)$$

The region of integration becomes a cylinder with an extremum condition for the maximum of the flux near one of its ends.

The boundary conditions are:

a) The condition for no flow through the wall of the nozzle $\theta|_{y=1} = \theta_K$; (4)

b) The symmetry condition on the nozzle axis $\theta|_{y=0} = 0$; (4a)

c) The condition at infinity $\theta(y)|_{x=-\infty} = 0$. (5)

Eqs. (3) are of the elliptic type in the subsonic flow region. It is known [7] that analytic solutions can be obtained for elliptic equations. If the region of integration along the flow is bounded below by the limiting characteristic, the unknown functions (u , θ , ρ , T) can be approximated, on the basis of the well-known Weierstrass theorem, to any degree of accuracy by algebraic polynomials in two variables (the coordinate origin lies on the nozzle axis in the section $x = x_0$, where $\theta(y)$ can be taken with an accuracy sufficient for all practical calculations to be equal to zero):

$$\begin{aligned} \theta_1 &= \theta|_{x > x_1} = y \left[\theta_k + \sum_{(i)} \sum_{(j)} b_{ij} (y-1)^j (x-x_1)^i \right], \\ \theta_2 &= \theta|_{x < x_1} = y \left[\theta_k + \sum_{(i)} \sum_{(j)} \beta_{ij} (y-1)^j x^i \right], \\ u &= \sum_{(i)} \sum_{(j)} a_{ij} y^{j-1} x^i \quad (i=0, 1, 2, \dots, p; j=1, 3, 5, \dots, q), \\ T &= \left[1 - \frac{n-1}{2} u^2 (1 + \theta^2) \right] + \sum_{(s)} \sum_{(l)} t_{sl} y^l x^s, \\ \rho &= \left[1 - \frac{n-1}{2} u^2 (1 + \theta^2) \right]^{\frac{1}{n-1}} + \sum_{(s)} \sum_{(l)} r_{sl} y^l x^s \quad (s=0, 2, \dots; l=1, 3, \dots). \end{aligned} \quad (6)$$

Here, x_1 is the abscissa of the point where the rim of the chamber makes contact with the output part of the nozzle (see Fig. 1).

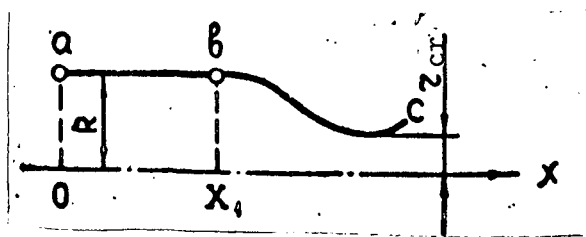


Fig. 1. Nozzle contour.

The form which is selected to represent T and ρ must be such that the solution process is stable.

The polynomials for θ , when $x = x_1$, are matched using the conditions:

$$\theta_1|_{x=x_1} = \theta_2|_{x=x_1}; \quad \left(\frac{\partial \theta_1}{\partial x} \right)_{x=x_1} = \left(\frac{\partial \theta_2}{\partial x} \right)_{x=x_1}.$$

The boundary conditions can be satisfied exactly by choosing appropriately the coefficients of the polynomial for θ .

The formulation of the last boundary condition, for setting up a flow regime with maximum flux in the throat, requires additional explanation.

In the beginning, of the solution the distribution of the flow parameters in the cross section of the nozzle in the region of integration is unknown.

The absolute maximum mass flux q_{\max} is attained in the throat of the nozzle if the flow is uniform in the section $x = x_{cr}$. For any arbitrary form of the contour of the subsonic part of the nozzle

$$\max q|_{x=x_{cr}} < q_{\max}.$$

To satisfy the condition for the maximum flux in the throat, /63 it is necessary to solve the resulting variational problem of

maximizing the functional (2) with the nonholonomic relations (3). The field of flow parameters in the subsonic part on which the functional (2) attains the maximum and the boundary conditions (4), (5) are satisfied is the solution of the problem which was formulated. The variational problem which was formulated can be solved, for example, using the well-known method of Lagrange multipliers [6]. However, when the latter is used, it is necessary to determine these multipliers in the solution, which increases the number of computations. Therefore, the solution was obtained differently.

In the class of functions (6), the problem which was formulated can be reduced to the problem of finding the maximum of a function of a single variable coefficient ϕ_k , if the remaining coefficients are expressed in terms of this coefficient on the basis of the system of equations (3) and the boundary conditions (4), (5). Then (2) attains a maximum when the relations

$$\frac{dq}{d\varphi_k} = \sum_{(i)} \frac{\partial q}{\partial \varphi_i} \cdot \frac{d\varphi_i}{d\varphi_k} = 0 \quad (i=0, 1, \dots, k-1, k, k+1, \dots, n). \quad (7)$$

are satisfied.

To determine the derivatives $\frac{d\phi_i}{d\phi_k}$, the information is used which is contained in the matrix of the system of linear equations for the method, which essentially is the Newton-Kantorovich method for the solution of nonlinear functional equations.

Suppose that an initial approximation for the values of the coefficients of the polynomials is known. The latter can be obtained, for example, by approximating the computational results

which were obtained in the one-dimensional approximation. We introduce the corrections $\Delta\phi_i$ for the values of the coefficients ϕ_i in such a way that in each successive approximation the coefficients of the polynomials will be calculated using the formula:

$$\boxed{\varphi_i^{(r+1)} = \varphi_i^{(r)} + \Delta\varphi_i} \quad (8)$$

Here, r is the number of the approximation.

Then the process of solving approximately the system of differential equations (3) can be replaced by the process of solving iteratively the system of improved equations

$$\boxed{\left(\frac{\partial F_g}{\partial \varphi_i}\right)_0 \Delta\varphi_i + F_{g,0} = 0 \quad (g=1, \dots, 4)} \quad (9)$$

using the Newton method on a finite set N of points in the integration region.

The system of these equations is an overdetermined system of linear algebraic equations in the corrections $\Delta\phi_i$.

We require that corrections be found which satisfy Eq. (9) in the best way (in the sense of the sum of squares of deviations) at the points in the region under consideration (the least squares method). This is achieved when the following equations are satisfied:

$$\boxed{\frac{\partial}{\partial \varphi_i} \sum_{m=1}^N \left[\left(\frac{\partial F_g}{\partial \varphi_i} \right)_0 \Delta\varphi_i + F_{g,0} \right]^2 = 0.}$$

As a result, we obtain a system of linear algebraic equations from which the corrections $\Delta\phi_i$ are determined. The augmented matrix of this system has the form

$\Delta\varphi_1$	$\Delta\varphi_2$	$\Delta\varphi_3$...	$\Delta\varphi_{n-2}$	$\Delta\varphi_{n-1}$	$\Delta\varphi_n$	Free term
A_{11}	A_{12}	A_{13}	...	$A_{1, n-2}$	$A_{1, n-1}$	$A_{1, n}$	B_1
A_{21}	A_{22}	A_{23}	...	$A_{2, n-2}$	$A_{2, n-1}$	$A_{2, n}$	B_2
A_{31}	A_{32}	A_{33}	...	$A_{3, n-2}$	$A_{3, n-1}$	$A_{3, n}$	B_3
...
$A_{n-2, 1}$	$A_{n-2, 2}$	$A_{n-2, 3}$...	$A_{n-2, n-2}$	$A_{n-2, n-1}$	$A_{n-2, n}$	B_{n-2}
$A_{n-1, 1}$	$A_{n-1, 2}$	$A_{n-1, 3}$...	$A_{n-1, n-2}$	$A_{n-1, n-1}$	$A_{n-1, n}$	B_{n-1}
$A_{n, 1}$	$A_{n, 2}$	$A_{n, 3}$...	$A_{n, n-2}$	$A_{n, n-1}$	$A_{n, n}$	B_n

(A)

The matrix (A) is used to find the derivatives in expression (7). Interchanging the columns for the coefficients of $\Delta\phi_k$ and $\Delta\phi_n$, and applying Gauss' direct elimination method, we obtain the triangular matrix:

$\Delta\varphi_1$	$\Delta\varphi_2$...	$\Delta\varphi_n$...	$\Delta\varphi_{n-2}$	$\Delta\varphi_{n-1}$	$\Delta\varphi_k$	Free term
<u>1</u>	C_{12}	...	$C_{1, n}$...	$C_{1, n-2}$	$C_{1, n-1}$	$C_{1, k}$	D_1
	<u>1</u>	...	C_{2n}	...	$C_{2, n-2}$	$C_{2, n-1}$	$C_{2, k}$	D_2
		...	\vdots	...	\vdots	\vdots	\vdots	\vdots
			<u>1</u>	...	$C_{k, n-2}$	$C_{k, n-1}$	$C_{k, k}$	D_k
				...	\vdots	\vdots	\vdots	\vdots
					<u>1</u>	$C_{n-2, n-1}$	$C_{n-2, k}$	D_{n-2}
						<u>1</u>	$C_{n-1, k}$	D_{n-1}
							<u>1</u>	D_n

(B)

Clearly, $\frac{d(\Delta\varphi_{n-1})}{d(\Delta\varphi_k)} = C_{n-1, k}$

$$\frac{d(\Delta\varphi_{n-2})}{d(\Delta\varphi_k)} = C_{n-2, k} + C_{n-2, n-1}(C_{n-1, k}) \text{ etc.}$$

i.e. the derivatives $d(\Delta_{\phi 1})/d(\Delta_{\phi k})$ can be found by the backward elimination method, if the column of coefficients of $\Delta_{\phi k}$ is transferred with opposite sign into the column of free terms, and the elements of the k-th column are set equal to zero, except the last, which is set equal to 1. In this case, the augmented matrix will have the form

$\frac{d(\Delta \varphi_1)}{d(\Delta \varphi_k)}$	$\frac{d(\Delta \varphi_2)}{d(\Delta \varphi_k)}$	$\frac{d(\Delta \varphi_3)}{d(\Delta \varphi_k)}$...	$\frac{d(\Delta \varphi_{n-2})}{d(\Delta \varphi_k)}$	$\frac{d(\Delta \varphi_{n-1})}{d(\Delta \varphi_k)}$	$\frac{d(\Delta \varphi_k)}{d(\Delta \varphi_k)}$	$\Delta \varphi_k$
1	C_{1k}	C_{2k}	...	$C_{1, n-1}$	$C_{2, n-1}$	0	$C_{1, k}$
	1	C_{3k}	...	$C_{2, n-1}$	$C_{3, n-1}$	0	$C_{2, k}$
		1	...	$C_{3, n-1}$	$C_{4, n-1}$	0	$C_{3, k}$
		
				1	$C_{n-1, n-1}$	0	$C_{n-1, k}$
					1	1	$C_{n-1, k}$

(C)

Matrix (C) corresponds to a linear system of algebraic equations for determining the derivatives $\frac{d(\Delta_{\phi 1})}{d(\Delta_{\phi k})}$.

But according to (8)

$$\frac{d(\Delta \varphi_l)}{d(\Delta \varphi_k)} = \frac{d \varphi_l}{d \varphi_k}.$$

Thus, it is possible to find the derivatives of all

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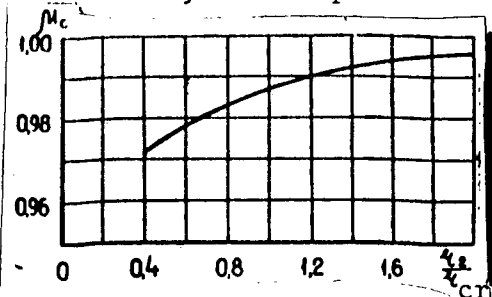


Fig. 2. Velocity distribution of the flow along the length of the nozzle.

coefficients which are necessary for the computations from the coefficient which was taken as the independent coefficient. This means that the variational problem for the extremum of the functional (2) was reduced to a problem of

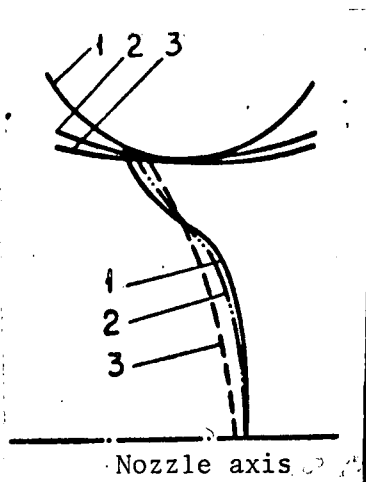


Fig. 3. Change in the form of the surface in the transition through the sonic velocity vs. r_2 .
1 - Contour of the nozzle and sound line for the nozzle with $\bar{r}_2 = 0.5$,
2 and 3 - same for nozzles with $\bar{r}_2 = 1$ and $\bar{r}_2 = 2$.

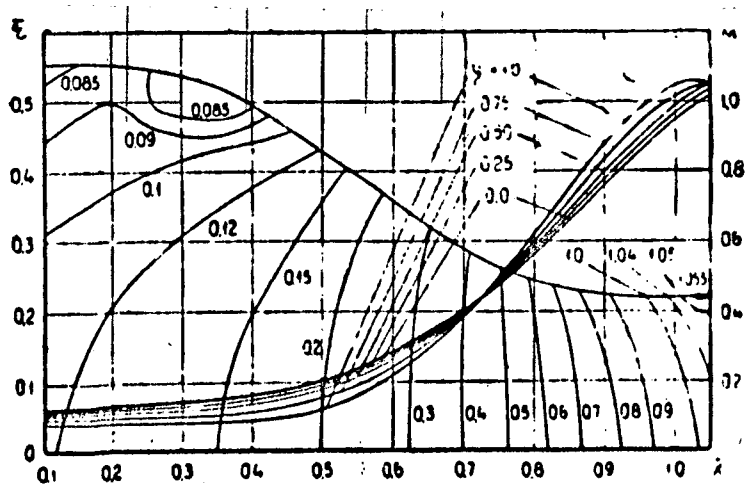


Fig. 4. Change in discharge coefficient of nozzles vs. rounding radius of the profile in the throat.

finding the extremum of a function of a single variable, a problem whose solution presents no difficulties.

Eq. (7), after the definite integral in (2) is replaced by Simpson's formula, gives a non-linear algebraic equation which can be used to determine the independent coefficient. This coefficient can be any coefficient in the polynomials (6).

Using the sequence of operations which was described, it is possible to find polynomials which will satisfy the boundary condition exactly, and which will satisfy the differential equations in the best way, in the mean square deviation sense.

The algorithm for solving the direct flow problem in the nozzle with the given form of the contour was coded as a program for the "BESM-6" electronic digital computer. The computations which were carried out

for a series of nozzles demonstrated the high accuracy of the method. The accuracy of the solution was checked using the integral rate equation. Computations have shown that the rate error in the region of the nozzle throat does not exceed 0.2% and increases somewhat (up to 2%) as the distance from the throat of the nozzle to the depth of the chamber increases. The computer time needed to solve the problem using the program does not exceed 30 sec.

Figs. 2-4 give characteristic computational results.

The flow in the supersonic part of the nozzle is calculated conveniently using the method of characteristics, using as the initial conditions the results which were obtained with the aid of the method presented above.

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¹ [Translator's note: 'Evidently means two-phase']